Addendum to Lecture 7

Another derivation of the uncertainty relations

Here we present an alternative derivation of the uncertainty relation,

$$\Delta A_\psi \Delta B_\psi \geq \frac{1}{2} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|,$$  \hspace{1cm} (1)

which do not make use of the anticommutator \{\hat{A}, \hat{B}\} ≡ \hat{A}\hat{B} + \hat{B}\hat{A} of the two operators \hat{A} and \hat{B}. Our basic assumptions are:

(a) The wave function \(\psi(q) \equiv |\psi\rangle\) is normalized as \(\langle \psi | \psi \rangle = 1\) (the symbol “\(\equiv\)” stands for “is represented by”). Here and hereafter the scalar products of the form \(\langle \phi | \hat{O} | \psi \rangle\), where \(\hat{O}\) is any operator (if \(\hat{O} = \hat{I}\), then \(\langle \phi | \hat{O} | \psi \rangle = \langle \phi | \psi \rangle\)), are intended to be calculated as:

$$\langle \phi | \hat{O} | \psi \rangle \equiv \int \phi^\ast(q) \hat{O}\psi(q) dq.$$  \hspace{1cm} (2)

(b) The operators \(\hat{A}\) and \(\hat{B}\) have not to be Hermitian. This implies that their expectation values in the state \(|\psi\rangle\) are not necessarily real numbers: \(\langle \hat{A} \rangle_\psi, \langle \hat{B} \rangle_\psi \in \mathbb{C}\), where \(\langle \hat{A} \rangle_\psi \equiv \langle \psi | \hat{A} | \psi \rangle\) and \(\langle \hat{B} \rangle_\psi \equiv \langle \psi | \hat{B} | \psi \rangle\).

(c) The standard deviation (square root of the variance) of the operator \(\hat{A}\) in the state \(|\psi\rangle\) is defined as \(\Delta A_\psi \equiv \sqrt{\langle \psi | \hat{A}^\dagger \hat{A} | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2}\), where \(\hat{A}_\psi \equiv \hat{A} - \langle \hat{A} \rangle_\psi\). A similar relations holds for \(\hat{B}\).

(d) For the sake of clarity, in the remainder we shall frequently use the shorthand \(\hat{a}\) for \(\hat{A}_\psi\) and \(\hat{b}\) for \(\hat{B}_\psi\).

To begin with, let \(\lambda \in \mathbb{R}\) be a real number and define the new wave function

\(|\phi(\lambda)\rangle \equiv (\hat{a} + i\lambda \hat{b}) |\psi\rangle\). \hspace{1cm} (3)

Then, by definition,

$$y(\lambda) \equiv ||\phi(\lambda)||^2 = \langle \phi(\lambda) | \phi(\lambda) \rangle \geq 0,$$  \hspace{1cm} (4)

for all \(\lambda \in \mathbb{R}\). A straightforward calculation shows that

$$y(\lambda) = \langle \psi | (\hat{a}^\dagger - i\lambda \hat{b}^\dagger)(\hat{a} + i\lambda \hat{b}) | \psi \rangle$$

$$= \langle \psi | \hat{b}^\dagger \hat{b} | \psi \rangle \lambda^2 + \langle \psi | i (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}) | \psi \rangle \lambda + \langle \psi | \hat{a}^\dagger \hat{a} | \psi \rangle$$

$$= (\Delta B_\psi)^2 \lambda^2 + \langle \psi | i (\hat{A}_\psi^\dagger \hat{B}_\psi - \hat{B}_\psi^\dagger \hat{A}_\psi) | \psi \rangle \lambda + (\Delta A_\psi)^2.$$  \hspace{1cm} (5)
Since the operator \( i (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}) \) is Hermitian by definition, then \( y(\lambda) \) is an ordinary second-degree polynomial with real coefficients in the variable \( \lambda \). The condition \( y(\lambda) \geq 0 \) is evidently satisfied if and only if the solutions of the quadratic equation \( y(\lambda) = 0 \) are complex numbers (see figure below).

These solutions are

\[
\lambda_{\pm} = \frac{1}{2\beta^2} \left( -\gamma \pm \sqrt{\gamma^2 - 4\beta^2\alpha^2} \right),
\]

where we have defined the three real numbers

\[
\beta^2 \equiv (\Delta B_\psi)^2, \quad \gamma \equiv \langle \psi | i (\Delta A_\psi \Delta B_\psi - \Delta B_\psi \Delta A_\psi) | \psi \rangle, \quad \alpha^2 \equiv (\Delta A_\psi)^2.
\]

Therefore, \( y(\lambda) \geq 0 \) if and only if \( \gamma^2 - 4\beta^2\alpha^2 \leq 0 \), namely if

\[
(\Delta A_\psi)^2 (\Delta B_\psi)^2 \geq \frac{1}{4} \langle \psi | i (\Delta A_\psi \Delta B_\psi - \Delta B_\psi \Delta A_\psi) | \psi \rangle^2
= \frac{1}{4} \left| \langle \psi | \Delta A_\psi^\dagger \Delta B_\psi - \Delta B_\psi^\dagger \Delta A_\psi | \psi \rangle \right|^2.
\]

If \( \hat{A} \) and \( \hat{B} \) were Hermitian operators, then this inequality would reduce to the more familiar one:

\[
\Delta A_\psi \Delta B_\psi \geq \frac{1}{2} \left| \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \right|.
\]

When the inequality (8) becomes an equality? Evidently, from (8) it follows that if either

\[
\Delta A_\psi | \psi \rangle = (\hat{A} - \langle \hat{A} \rangle_\psi) | \psi \rangle = 0, \quad \text{or} \quad \Delta B_\psi | \psi \rangle = (\hat{B} - \langle \hat{B} \rangle_\psi) | \psi \rangle = 0,
\]

then both sides are equal to zero and (8) reduces to the trivial identity \( 0 = 0 \). This is not surprising because equations (10) imply that \( | \psi \rangle \) is a common wavefunction of \( \hat{A} \) and \( \hat{B} \) with eigenvalues \( \langle \hat{A} \rangle_\psi \) and \( \langle \hat{B} \rangle_\psi \), respectively. Therefore, we have \( \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle = 0 \).
The second possibility occurs when \( \gamma^2 - 4 \beta^2 \alpha^2 = 0 \) and the two solutions coincides: \( \lambda_+ = \lambda_- \equiv \lambda_0 \), where
\[
\lambda_0 = \frac{\gamma}{2\beta^2}
\]
\[
= -\frac{\langle \psi | i (\hat{\Delta}^\dagger A^\dagger \hat{\Delta}^\dagger B \psi - \hat{\Delta}^\dagger B \hat{\Delta}^\dagger A \psi) | \psi \rangle}{2 \langle \psi | \hat{\Delta}^\dagger B \hat{\Delta}^\dagger B \psi | \psi \rangle}.
\] (11)

This occurrence is illustrated by the green parabola in the figure above. In this case we have \( y(\lambda_0) = 0 \), which means \( ||\phi(\lambda_0)||^2 = 0 \). However, we know that the norm of a vector is zero only when the vector itself is null, that is \( |\phi(\lambda_0)| = 0 \). Therefore, from (5) we infer that inequality (8) becomes an equality only for those special wavefunctions \( |\psi\rangle \) that are solutions of the equation,
\[
(\hat{a} + i \lambda_0 \hat{b}) |\psi\rangle = 0.
\] (12)

Using the previous results, this can be rewritten as:
\[
(\hat{A} - \langle \hat{A} \rangle_\psi) |\psi\rangle = i \frac{\langle \psi | i (\hat{\Delta}^\dagger A^\dagger \hat{\Delta}^\dagger B \psi - \hat{\Delta}^\dagger B \hat{\Delta}^\dagger A \psi) | \psi \rangle}{2 \langle \psi | \hat{\Delta}^\dagger B \hat{\Delta}^\dagger B \psi | \psi \rangle} (\hat{B} - \langle \hat{B} \rangle_\psi) |\psi\rangle.
\] (13)

It is instructive to study the form of the right side of (8). A long but straightforward calculation shows that
\[
\langle \psi | \hat{\Delta}^\dagger A^\dagger \hat{\Delta}^\dagger B \psi - \hat{\Delta}^\dagger B \hat{\Delta}^\dagger A \psi | \psi \rangle = \langle \psi | \hat{A}^\dagger \hat{B} \psi \rangle - \langle \psi | \hat{A}^\dagger \psi \rangle \langle \psi | \hat{B} \psi \rangle
- \langle \psi | \hat{B}^\dagger \hat{A} \psi \rangle + \langle \psi | \hat{B}^\dagger \psi \rangle \langle \psi | \hat{A} \psi \rangle.
\] (14)

If \( \hat{A} = \hat{A}^\dagger \) and \( \hat{B} = \hat{B}^\dagger \), then \( \langle \psi | \hat{A}^\dagger \psi \rangle = \langle \psi | \hat{A} \psi \rangle \) and \( \langle \psi | \hat{B}^\dagger \psi \rangle = \langle \psi | \hat{B} \psi \rangle \). This simplifies (14) to
\[
\langle \psi | \hat{\Delta}^\dagger A^\dagger \hat{\Delta}^\dagger B \psi - \hat{\Delta}^\dagger B \hat{\Delta}^\dagger A \psi | \psi \rangle = \langle \psi | \hat{A} \hat{B} \psi \rangle.
\] (15)

Now, it should be remembered that the covariance of two random variables, say \( x \) and \( y \), is defined as:
\[
cov [x, y] = E[xy] - E[x] E[y],
\] (16)

where \( E[x] \) denotes the expectation value of \( x \). Using this notation, we could rewrite (14) as:
\[
\langle \psi | \hat{\Delta}^\dagger A^\dagger \hat{\Delta}^\dagger B \psi - \hat{\Delta}^\dagger B \hat{\Delta}^\dagger A \psi | \psi \rangle = \cov \psi [\hat{A}^\dagger, \hat{B}] - \cov \psi [\hat{B}^\dagger, \hat{A}].
\] (17)
What is the meaning of these uncertainty relations when $\hat{A}$ and $\hat{B}$ are both Hermitian, so they represent measurable physical quantities (“observables”, for short)? Quoting Galindo&Pascual\textsuperscript{1},

The uncertainty relation tells us that, in general, it is impossible to prepare systems in such a way that one can simultaneously measure $\hat{A}$ and $\hat{B}$ with arbitrarily small standard deviations, unless $[\hat{A}, \hat{B}] = 0$. If $[\hat{A}, \hat{B}] \neq 0$ and one prepares the system so that $\Delta A\psi$ has a given value, then $\Delta B\psi$ must satisfy relation (1); thus, in general, diminishing $\Delta A\psi$ increases $\Delta B\psi$ in such a way that (1) is always satisfied. We could say that the uncertainty relation expresses the alteration of the potentiality of the $\hat{A}$ values in $|\psi\rangle$ as a result of the process of actualizing the $\hat{B}$ values through measurement.

Another interesting comment is from Peres\textsuperscript{2}:

An uncertainty relation such as (1) is not a statement about the accuracy of our measuring instruments. On the contrary, its derivation assumes the existence of perfect instruments (the experimental errors due to common laboratory hardware are usually much larger than these quantum uncertainties). The only correct interpretation of (1) is the following: If the same preparation procedure is repeated many times, and is followed either by a measurement of $\hat{A}$, or by a measurement of $\hat{B}$, the various results obtained for $\hat{A}$ and for $\hat{B}$ have standard deviations, $\Delta A\psi$ and $\Delta B\psi$, whose product cannot be less than $|\langle[\hat{A}, \hat{B}]\rangle\psi|/2$. There never is any question here that a measurement of $\hat{A}$ “disturbs” the value of $\hat{B}$ and vice-versa, as sometimes claimed. These measurements are indeed incompatible, but they are performed on different particles (all of which were identically prepared) and therefore these measurements cannot disturb each other in any way. The uncertainty relation (1), [...], only reflects the intrinsic randomness of the outcomes of quantum tests.

\textsuperscript{1}A. Galindo, P. Pascual, Quantum Mechanics, Vol. I, pp. 54-55 (Springer, 1990)
\textsuperscript{2}A. Peres, Quantum Theory: Concepts and methods, p. 93 (Kluver Academic Publishers, 1995)