Entanglement of mechanical oscillators coupled to a nonequilibrium environment

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Recent experiments aim at cooling nanomechanical resonators to the ground state by coupling them to nonequilibrium environments in order to observe quantum effects such as entanglement. This raises the general question of how such environments affect entanglement. Here we show that there is an optimal dissipation strength for which the entanglement between two coupled oscillators is maximized. Our results are established with the help of a general framework of exact quantum Langevin equations valid for arbitrary bath spectra, in and out of equilibrium. We point out why the commonly employed Lindblad approach fails to give even a qualitatively correct picture.

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I. INTRODUCTION

Entanglement [1] constitutes a cornerstone of quantum mechanics and is a major subject of present-day research [2]. Whether it persists and can be observed in systems comprising macroscopic bodies has been a hotly debated topic since the early days of quantum mechanics. The ground state of two interacting quantum systems will generally be entangled. Thus, one could naively expect that it is sufficient to simply cool two interacting, macroscopic bodies to their ground states and thereby prepare an entangled state. However, when coupling to a dissipative bath—as is of course necessary for cooling—entanglement may be destroyed, as explored in a number of works, for example, [3]. A slate of recent experiments has now brought a new aspect into focus: A nonequilibrium environment, consisting of either a driven optical cavity [4], a superconducting microwave resonator [5], or a superconducting single-electron transistor [6], can be employed to cool the motion of mechanical resonators down to the ground state. The advances in this field may ultimately enable tests of quantum mechanics in an entirely new regime [7] and to observe entanglement of massive objects [8,9]. Still it remains to resolve the issue of how the dissipative coupling to the nonequilibrium bath affects entanglement.

In the present work, we demonstrate a nonmonotonic dependence of entanglement between two oscillators on the coupling strength to the nonequilibrium environment and show that there is an optimal value for the coupling to the bath. Below this value, entanglement is diminished by thermal fluctuations, and above this value, it is lost through dissipation. The striking behavior found here is missed entirely by the commonly employed Lindblad approach to dissipative dynamics.

In order to obtain an exact description, we develop a general framework based on quantum Langevin equations, which allows us to analyze the entanglement between harmonic oscillators in the presence of coupling to a linear bath of arbitrary spectral density. First, we exploit this scheme to show that even in equilibrium there are effects likely to be missed by simpler approaches. For example, the minimum coupling strength needed for entanglement depends logarithmically on the cutoff frequency for the most important case of an Ohmic bath spectrum. For the case of a nonequilibrium bath, we illustrate the generic behavior in a concrete example of two mechanical resonators inside an optical cavity, being cooled by the optomechanical interaction with the light field circulating in the cavity.

II. MODEL

We consider two coupled oscillators with masses \(m_{A,B}\) and frequencies \(\Omega_{A,B}\) (see Fig. 1). In terms of their positions and momenta, \(\hat{q}_{A,B}\) and \(\hat{p}_{A,B}\), the Hamiltonian reads \(\hat{H}_{\text{sys}} = \sum_{\alpha=A,B} m_{\alpha} \hat{q}_{\alpha}^{2}/2 + \frac{\hat{p}_{\alpha}^{2}}{2m_{\alpha}} + k(\hat{q}_{A} - \hat{q}_{B})^{2}/2\), with a coupling spring constant \(k\). Moreover, we assume the oscillators to be subject to fluctuating quantum forces \(\hat{F}_{\alpha}\), which are possibly correlated, and which derive from a bath of harmonic oscillators, with \(\hat{H}_{\text{sys-bath}} = \sum_{\alpha} \hat{q}_{\alpha} \hat{F}_{\alpha}\). They will be characterized by their spectra as specified below.

If the state of the environment is Gaussian, the oscillators also end up in a Gaussian state, which is fully described by the covariance matrix \(\gamma_{ij} = \text{tr}(\hat{\rho} (\hat{R}_{i} \hat{R}_{j})/2)\). Here \(\hat{R} = (\hat{F}_{A}, \hat{q}_{A}, \hat{p}_{A}, \hat{q}_{B}, \hat{p}_{B})^{T}\), \(\langle \hat{R} \rangle \equiv 0\) in steady state, and \(\hat{\rho}\) is the system’s density matrix. As a measure of the entanglement between the oscillators, the logarithmic negativity [10–12] is calculated as \(E_{N}(\hat{\rho}) = \sum_{i=1,2} \frac{1}{2} f(\xi_{i})\), where \(f(\xi) = -\log_{2}(2\xi)\) for \(\xi < 0.5\) and \(f(\xi) = 0\) otherwise, and where \(\xi_{1,2}\) are the symplectic eigenvalues of the partially transposed covariance matrix \(\gamma^{T}\) [12].

For later use, and in order to fix the notation, it will be convenient to consider first the simple example of two identical oscillators \((m_{A/B} = m, \Omega_{A/B} = \Omega)\) at thermal equilibrium and assume the coupling to the environment to be negligible. The system can be decoupled by introducing the normal-mode coordinates \(\hat{\eta}_{\pm} = (\hat{q}_{A} \pm \hat{q}_{B})/\sqrt{2}\) and momenta \(\hat{p}_{\pm} = (\hat{p}_{A} \pm \hat{p}_{B})/\sqrt{2}\) corresponding to the center-of-mass motion \((\Omega_{c} = \Omega)\) and the relative motion at frequency
oscillators ($\omega_1$ for the Heisenberg operators, obtained by
destroy entanglement. It is well known, thermal fluctuations will reduce and eventually
and $\langle F_a F_b \rangle_0 = \sum_{\alpha=\pm} \chi_{\alpha \beta}(\omega_0) \langle \hat{q}_\alpha \hat{q}_\beta (\omega_0) \rangle$.
Here $\langle F_a F_b \rangle_0 = \int dt e^{i \omega t} \langle \hat{F}_a (t) \hat{F}_b (0) \rangle$ and $\chi_{\alpha \beta}(\omega_0)$
are elements of a matrix whose inverse is given by
$[\chi^{1-1}(\omega_0)]_{\alpha \beta} = m_{\alpha \beta} (\omega_0^2 - \omega^2) + k - \chi_{\alpha \beta}(\omega_0)$
and $[\chi^{1-1}(\omega_0)]_{\alpha \beta} = -k - \chi_{\alpha \beta}(\omega_0)$ for $\alpha \neq \beta$.

Momentum correlators follow from
$\langle \hat{p}_a \hat{p}_b \rangle_0 = \int_0^\infty dt e^{i \omega t} \langle \hat{q}_a (t) \hat{q}_b (0) \rangle$.
Finally, equal-time correlators are obtained by integration, $\langle \hat{q}_a \hat{q}_b \rangle_0 = \int_0^\infty dt e^{i \omega t} \langle \hat{q}_a \hat{q}_b \rangle_0$.
The solution of (1) thus provides the full covariance matrix
$\chi_2(\omega)$ of the thermal fluctuations between two
coherent oscillators. Note that we did not use
the quantum entropy, that is, the fluctuations dissipation relation
between $\chi_2(\omega)$ and $\langle F_a F_b \rangle_0$ does not necessarily hold.

For simplicity, we will from now on restrict our explicit
calculations to the symmetric case of two identical oscillators that
are connected to two independent baths, such that
$\langle \hat{F}_a \hat{F}_b \rangle_0 = \delta_{a \beta} \langle \hat{F}_0 \rangle_0$ and $\chi_2(\omega) = \delta_{a \beta} \chi^2(\omega)$.
The system can then, as before, be decomposed into the center-of-mass
state $|\hat{q}_+\rangle$, $|\hat{q}_-\rangle$, and the relative mode $|\hat{q}_\pm\rangle$, which
become independent dissipative oscillators. We find
$\langle \hat{q}_+ \hat{q}_\pm \rangle_0 = \langle \hat{F}_0 \rangle_0 |\chi_{\pm}(\omega)|^2$, where $\chi_{\pm}(\omega) = |m \Omega_0^2 - \omega^2 - \chi^2(\omega)|^{-1/2}$.
After frequency integration, Eq. (1) thus directly yields the
logarithmic negativity.

IV. EQUILIBRIUM BATH

First, we illustrate the general scheme for the case of
equilibrium baths, picking the important example of an Ohmic
bath spectrum: $\langle \hat{F}_a \hat{F}_a \rangle_0 = \langle \hat{F}_0 \rangle_0 = m \Gamma_m \omega_0 \coth(\omega_0/2T) + 1/(1 + \omega^2/\omega_c^2)$. Here $T$ denotes the temperature, $\Gamma_m$ the
damping rate, and $\omega_c$ the cutoff frequency. For $\Gamma_m < \Omega$ and
$\omega_c > \Omega$, the position and momentum variances of an oscillator
coupled to this bath are given analytically in [14]. Here we only
display the expansion to first order in $\Gamma_m / \Omega$ at $T = 0$:

$$2m \Omega_0^2 \langle \hat{q}_\pm^2 \rangle_0 \approx 1 - \frac{\Gamma_m}{\pi \Omega_\pm},$$

$$2 \langle \hat{q}_\pm^2 \rangle_0 / m \Omega_\pm \approx 1 + \frac{\Gamma_m}{\pi \Omega_\pm} \left[ 2 \ln \frac{\omega_c}{\Omega_\pm} - 1 \right].$$

As illustrated in Fig. 2, entanglement between the oscillators is suppressed due to their coupling to the bath. The high-frequency bath modes cause momentum fluctuations that depend logarithmically on the cutoff frequency [cf. Eq. (3)]. Thus, even at zero temperature, the coupling to the environment reduces the logarithmic negativity by $\Gamma_m [\ln(\omega_c / \Omega) - 1]/(\pi \Omega \ln 2)$ [as follows from Eqs. (1) and (3)], and eventually destroys the entanglement completely. Entanglement persists ($E_N > 0$) only if the coupling rate exceeds a threshold value

$$G_{\text{Ohmic}, T=0}^{\text{threshold}} = \frac{\Gamma_m}{\pi} \left( \ln \frac{\omega_c}{\Omega} - 1 \right).$$

As a distinctive feature, the minimal coupling rate depends
logarithmically on the cutoff frequency. It indicates that any approach that disregards the influence of high-frequency
fluctuations has to fail, as discussed for the example of the Lindblad approach below. Our general formula also allows to obtain the full temperature-dependence (see Fig. 2).

V. NONEQUILIBRIUM BATH

Tunable nonequilibrium quantum fluctuations are now relevant in many contexts and may be used, for example, to cool systems below the bulk temperature. A paradigmatic example is the photon shot noise coupled to mechanical resonators in optomechanical setups [4,15] (the following results also apply to analogous electromechanical systems [5,6]). We treat the conceptually clearest case where two nanomechanical membranes are placed inside a laser-driven cavity, and two independent light forces $F_{\pm}^{cav}$ act on the mechanical normal modes $\hat{n}_{\pm}$, leading to optomechanical cooling [16,17]. This may be realized in a setup with two cavity modes, where $\hat{H}_{\text{sys-cav}} = (g/\ell_m)[(\hat{a}_+ + \hat{a}_-\hat{n}_+) + (\hat{a}_+ + \hat{a}_-\hat{n}_-)]$ (see Fig. 3). Here $\hat{a}_\pm$ are the annihilation operators of the cavity modes, $\ell_m = 1/\sqrt{2m\Omega}$ is the mechanical ground-state width, and $g$ is the oscillator-cavity coupling rate that scales linearly with the laser amplitude (see [18,19] for a derivation of this type of coupling). The mechanical coupling $k$ between the oscillators (here assumed as given) can itself be implemented via other, strongly driven far-detuned cavity modes [9,18]. Other possible setups include cold-atom or hybrid atom-membrane systems [18].

Elimination of the cavity degrees of freedom generates cavity noise spectra [16] $\{\hat{F}_{\pm}\hat{F}_{\pm}^{\dagger}\}^{cav} = (g/\ell_m)^2 \kappa [(o + \Delta_{\pm})^2 + \kappa^2/4]^{-1}$, where $\kappa$ is the decay rate of the cavity photons and $\Delta_{\pm}$ the detuning of the corresponding input lasers with respect to the first (second) cavity mode. A spectrum of this kind induces an optomechanical cooling rate of $\Gamma_{\text{opt}} = k^2/(\ell_m^2\kappa) = (\hat{F}_{\pm}\hat{F}_{\pm}^{\dagger})^{cav}/\Omega_{\pm}$. In the optimal cooling regime, for $\Delta_{\pm} = -\Omega_{\pm}$ and $\kappa \ll \Omega$, we have $\Gamma_{\text{opt}} \approx \Gamma_{\text{opt}}^0 = 4g^2/k$. In this regime, the minimum possible phonon number due to optical cooling, defined by $(n_{\text{opt}} + 1)/n_{\text{th}} = (\hat{F}_{\pm}\hat{F}_{\pm}^{\dagger})_{cav}/(\hat{F}_{\pm}\hat{F}_{\pm}^{\dagger})_{th} - 1$, will be much smaller than $1 \left( n_{\text{opt}} \approx n_{\text{opt}}^0 = (\kappa/4\Omega_{\pm})^2 \right)$. Moreover, we assume $\Gamma_{\text{opt}} n_{\text{th}} \ll \Omega$, $\Gamma_{\text{opt}} \ll \Gamma_{\text{opt}}^0$, and $g \ll \Omega$ as required for ground-state cooling. The full forces $\hat{F}_{\pm} = \hat{F}_{\pm}^{cav} + \hat{F}_{\pm}^{\ell}$ also contain thermal fluctuations $\hat{F}_{\pm}^{\ell}$, independent from $\hat{F}_{\pm}^{cav}$. For low mechanical damping ($\Gamma_m \ll \Omega$), the spectrum of the thermal bath can be replaced by the values at the resonances, that is, $\langle \hat{F}_{\pm}\hat{F}_{\pm}^{\dagger}\rangle_{\text{th}} \to \langle \hat{F}_{\pm}\hat{F}_{\pm}^{\dagger}\rangle_{\text{th}=\text{att}} = \gamma g/(\Omega_{\pm})^2$. The general scheme yields the variances by integrating $\langle \hat{n}_{\pm}\hat{n}_{\pm}\rangle_{\omega} = \langle (\hat{F}_{\pm}\hat{F}_{\pm}^{\dagger})_{\omega} + (\hat{F}_{\pm}\hat{F}_{\pm}^{\dagger})_{\omega} \rangle/\chi_{\pm}(\omega)$. In the optimal cooling regime, the variances of the optomechanically damped system can be expressed in a compact way:

$$\begin{align*}
2\langle \hat{n}_{\pm}^2 \rangle/m\Omega_{\pm} &\approx 1 + 2(n_{\text{eff}} + \delta n), \\
2m\Omega_{\pm}\langle \hat{n}_{\pm} \rangle &\approx 2\langle \hat{n}_{\pm}^2 \rangle/m\Omega_{\pm} + g^2/\Omega_{\pm}^2,
\end{align*}$$

where $n_{\text{eff}} = \Gamma_{\text{opt}} n_{\text{th}}/\Gamma_{\text{opt}} + n_{\text{opt}}$ and $\delta n = \Gamma_{\text{opt}} n_{\text{th}}/\kappa$.

Together with Eq. (1), these formulas constitute our main result for entanglement in a system subject to optomechanical cooling. We now extract and discuss its main physical features. The first term on the right-hand side (RHS) of Eq. (5) describes the ground-state energy, and the second term takes account of the cooling mechanism: the thermal occupation is reduced to an effective phonon number $n_{\text{eff}}$. Thus, entanglement can in principle be created even for large bulk temperatures, $n_{\text{th}} > 1$, if the optomechanical damping rate $\Gamma_{\text{opt}}$ is sufficiently large. Since $\Gamma_{\text{opt}} = 4g^2/\kappa$, this can be achieved either by reducing the cavity linewidth $\kappa$ or by increasing the cavity-oscillator coupling rate $g$. However, we identify two processes that destroy entanglement for small $\kappa$ and large $g$, respectively. First, as known from [16], the cooling mechanism becomes less efficient in the strong-coupling regime $\Gamma_{\text{opt}} \sim \kappa$, where the contribution of $\delta n$ becomes appreciable. Second, for a large optomechanical coupling strength $g$, the low-frequency contributions of the photon shot noise induce an increase of the position variance [second term on the RHS of Eq. (6)]. This implies that strong correlations between the individual oscillators and the driven cavity lead to a destruction of entanglement between the oscillators.

As a consequence, entanglement depends nonmonotonically on the cavity linewidth $\kappa$ and the optomechanical damping rate $\Gamma_{\text{opt}}$ in the optimal cooling regime (see Figs. 4 and 5).
to observe entanglement. Note that Eq. (7) can be employed to optimize entanglement.

entangled systems, these are given by \( L_i \) [13]. For equilibrium baths, these are given by \( L_{\text{eq}}(\hat{\rho}) = (\Gamma_m/2)(n_{\pm} + 1)D[\hat{A} \pm] \) and \( L_{\text{eq}}(\hat{\rho}) = (\Gamma_m/2)n_{\pm} D[\hat{A} \pm] \), where \( D[\hat{A}] \) is the mechanical normal-mode annihilation operators. At zero temperature, the shortcomings of the Lindblad approach are most obvious: The system evolves into its ground state, whose

VI. SHORTCOMINGS OF THE LINDBLAD APPROACH

The crucial destruction of entanglement by strong dissipation is missed entirely by the commonly employed Lindblad master equation approach. Its general form is given by \( \dot{\hat{\rho}} = -i[\hat{H}_{\text{sys}}, \hat{\rho}] + \sum_i L_i(\hat{\rho}) \), where the influence of the bath is taken into account by Lindblad terms \( L_i \) [13]. For equilibrium baths, these are given by \( L_{\text{eq}}(\hat{\rho}) = (\Gamma_m/2)(n_{\pm} + 1)D[\hat{A} \pm] \) and \( L_{\text{eq}}(\hat{\rho}) = (\Gamma_m/2)n_{\pm} D[\hat{A} \pm] \), where \( D[\hat{A}] \) is the mechanical normal-mode annihilation operators. At zero temperature, the shortcomings of the Lindblad approach are most obvious: The system evolves into its ground state, whose entanglement is not reduced at all by the system-bath coupling. To treat the nonequilibrium case of Fig. 3 in the Lindblad approach, we have to consider four additional terms. \( L_{\text{eq}}(\hat{\rho}) = (\Gamma_m/2)n_{\pm} D[\hat{A} \pm] \) and \( L_{\text{eq}}(\hat{\rho}) = (\Gamma_m/2)n_{\pm} D[\hat{A} \pm] \), which take account of the decoherence via the cavity modes (see [19] for a detailed derivation). The steady-state variances of the normal modes follow as \( 2\eta_{\pm}/(\eta_{\pm}^2 - 2\eta_{\pm} \approx 2\eta_{\pm} + 1 \). This expression describes the cooling to an effective phonon number \( n_{\text{eff}} \) but fails to capture the loss of entanglement for strong optomechanical coupling (see the dashed curve in Fig. 4). The shortcomings of this approach can be understood by noting that the Born-Markov approximation, which assumes the bath to have a very short correlation time (no memory) and to be uncorrelated with respect to the system, does not hold in general for a nonequilibrium bath, as can be seen in our example.

VII. CONCLUSIONS AND OUTLOOK

The general exact framework introduced here can be employed to analyze the entanglement of oscillators under the influence of arbitrary bath spectra, among them nonequilibrium and tailored nonstandard spectral densities. As pointed out in this paper, the effects of tunable nonequilibrium environments promise rich physics to be explored in current experimental setups. The optomechanical setup investigated here is in fact just one of a rather large class of setups to which this work applies, and which also extends into the fields of electromechanics [5,6] and cold-atom physics [18]. We also note that completely different systems show similar entanglement production effects under nonequilibrium conditions, as has been explored in the case of coupled, driven qubits [20], atoms [21], and ions [22], or coupled double quantum dots [23].

In the quest to observe entanglement in dissipatively cooled optomechanical or nanoelectromechanical systems, the theory presented here serves as an essential guideline: It identifies viable parameter regimes for generating and optimizing entanglement between massive mechanical oscillators.

Recent works [19,24] have proposed an alternative way of generating entanglement in nanomechanical systems: By modulation of the coupling strength between the oscillators, the system can be parametrically driven into a nonequilibrium state which features entanglement even at relatively large temperatures. In a future work, the general framework introduced here can be employed to discuss the generation of entanglement in a parametrically driven system and to compare and connect the two approaches.

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